One-dimensional spin glasses, uniqueness and cluster properties

1988 J. Phys. A: Math. Gen. 21 1781

(http://iopscience.iop.org/0305-4470/21/8/014)

J. Phys. A: Math. Gen. **21** (1988) 1781-1786. Printed in the UK

One-dimensional spin glasses, uniqueness and cluster properties†

Aernout C D van Enter‡

Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland

Received 2 October 1987

Abstract. We discuss some recent results on the absence of phase transitions in one-dimensional spin-glass models with polynomially decaying interactions. We comment on the probabilistic aspects and on the notion of 'weak uniqueness'.

1. Introduction

Spin glasses are among the more fashionable models of statistical mechanics. The original problem (and name) comes from the attempt to describe magnetic atoms (like Fe) which are diluted in a not too high concentration (such as 5%) in a non-magnetic environment (like Au) and which interact via the long-range oscillating RKKY interaction.

The Hamiltonian this problem gives rise to is

$$H = -\sum_{i,j} \varepsilon_i \varepsilon_j \frac{\cos k_{\mathrm{F}}(i-j)}{|i-j|^3} s_i s_j \tag{1}$$

where the quenched disorder variables $\varepsilon_i = 0, 1$ describe the dilution and the s_i are spin variables.

Owing to the oscillating character of the cosine and the long-range character of the $1/|i-j|^3$ interaction, a particular spin can be subject to many competing forces from the other spins. The combination of randomness and 'frustration' is generally modelled by Edwards-Anderson (EA) models of the form (1)

$$H = -\sum_{i,j} \tilde{J}(i, j) s_i s_j \tag{2}$$

where the site-random Hamiltonian (1) is replaced by a bond-random Hamiltonian in which the $\tilde{J}(i, j)$ are independent random variables with a distribution which only depends on the distance $|i-j|$. Both short- and long-range EA models have been studied. They have been applied to many other areas, in particular to the theory of neural networks and to optimisation problems.

Spin glasses have attracted extensive interest among physicists (for some recent reviews, see [2-5]). Up till now, it has been very difficult to obtain mathematically rigorous results on the presumed low-temperature spin-glass phase (for some recent

† Presented at the conference on *Mathematical Problems in Statistical Mechanics* held at Heriot-Watt University on 3-5 August 1987.
‡ Permanent address: Department of Physics, Technion, 32000 Haifa, Israel.

0305-4470/88/081781+06$02.50

heuristic theories, see [6–8]). On the other hand, during the last years there have been a number of results about the region where there is no phase transition (high temperature or low dimension), despite the possible occurrence of Griffiths singularities [9], which prevents the thermodynamic quantities to be analytic. In this paper I will describe some of these results for long-range models, in particular in one dimension, and discuss some conceptual problems, such as the 'weak' against 'strong' uniqueness of the Gibbs state.

2. Results for long-range models

The models we consider have Hamiltonians

$$H = -\sum_{i,j\in Z} |i-j|^{-\alpha} J(i,j) s_i s_j \tag{3}$$

where the $J(i,j)$ are independent, identically distributed random variables. We use the symbol E for taking the average over the disorder variables $\{J(i,j)\}$. The $\{J\}$ distribution satisfies

$$EJ(i,j) = 0 \tag{4a}$$

and (for small t)

$$E\exp(tJ(i,j)) = \exp(\mathrm{O}(t^2)). \tag{4b}$$

If we have a boundary condition σ outside a volume Λ we write

$$H_{\Lambda,\sigma} = \sum_{i,j\in\Lambda} \tilde{J}(i,j) s_i s_j + \sum_{\substack{i\in\Lambda\\ j\notin\Lambda}} \tilde{J}(i,j) s_i \sigma_j \tag{5}$$

$$(\tilde{J}(i,j) \equiv |i-j|^{-\alpha} J(i,j)).$$

The free energy of a volume Λ, at inverse temperature β and boundary condition σ is

$$\beta F_{\Lambda,\sigma}(\{J\},\beta) = (-1/|\Lambda|) \ln \mathrm{tr}_\Lambda \exp(-\beta H_{\Lambda,\sigma}(\{J\}\{s\}_\Lambda)). \tag{6}$$

These models have been studied both heuristically and rigorously. Here we describe the rigorously known results. A heuristic treatment, which also gave predictions for critical behaviour as a function of α, was given in [10].

In particular, theorems 2 and 3 confirm the predictions made there for the occurrence of the paramagnetic phase. As a phase transition is predicted for $\alpha < 1$, theorem 2 is supposedly a close to optimal result.

The following results are known.

Theorem 1 [11–13]. If $\alpha > \frac{1}{2}$ and $\Lambda \to \infty$ in the sense of Fisher,

$$\lim_{\Lambda\to\infty} F_{\Lambda,\sigma} = \lim_{\Lambda\to\infty} EF_{\Lambda,\sigma} = f$$

exists, J-almost surely, and does not depend on the $\{J\}$ nor on the boundary condition σ, as long as σ is chosen independent of the $J(i,j)$.

Remark. A weak version of this result (convergence of the mean free energy) was proved in [14].

A stronger version, which weakens condition (4*b*), was recently proved by Zegarlinsky [15]. (He requests existence of moments up to fourth order of the $J(i, j)$. In fact, using his stability bound ([15], formula A6) and the subadditive ergodic theorem as in [12], the argument works even if only the second moment exists.)

This theorem is actually valid in any dimension (if $\alpha d > \frac{1}{2}$). The next theorem, however, is an essentially one-dimensional result.

Theorem 2. If $\alpha > 1$, J-almost surely we have the following.

(*a*) There is no phase transition 'in the weak sense'. In the thermodynamic limit the Gibbs state is pure (extremal Gibbs) and does not depend on the (non-random) boundary condition σ.

(b) The correlation functions calculated with respect to this Gibbs state decay with the same decay rate as the interaction.

(c) The free energy is a C^∞ function of temperature and magnetic field.

Remark 1. A weaker form of theorem 2(*a*) (absence of symmetry breaking) was essentially proven in [13] and shortly after in a different way in [16]. The full proof of uniqueness and the observation that the arguments give a 'weak sense' proof were given in [17]. The fact that weak uniqueness suffices for physics was discussed before in [18] (boundary conditions represent the experimental set-up, which does not depend on the sample). Weaker upper bounds on asymptotic correlation decay than given in 2(*b*) were given in [17] and (for vector spins) in [19]. In its present form the theorem appeared shortly after the Heriot-Watt conference in [20].

Remark. For the case $\alpha > \frac{3}{2}$, strong uniqueness (there is only one Gibbs state, whatever boundary conditions one prescribes) was proved in [21].

The C^∞ property and the asymptotic correlation decay were proved in [22]. For vector spins the (strong) absence of symmetry breaking and an upper bound on the asymptotic correlation decay were proved in [23, 24].

In the case $\alpha > \frac{1}{2}$ (in general dimension d, $\alpha d > \frac{1}{2}$), high-temperature results have been obtained by Fröhlich and Zegarlinsky [15, 25, 26].

Theorem 3. Let $\alpha > \frac{1}{2}$. Then there is $\beta_0 > 0$, such that for $0 \leq \beta < \beta_0$ J-almost surely,

(*a*) the Gibbs state is weakly unique;

(*b*) the correlation functions decay asymptotically at the same rate as the potential; and

(c) the free energy is C^∞.

Remark. Recently Fröhlich and Zegarlinsky have applied their methods to obtain a rigorous treatment of the high-temperature phase of the Sherrington-Kirkpatrick model [27]. This also has been done via different methods by Aizenman *et al* [28].

3. Some remarks about proofs; reduction to a non-random problem and weak against strong uniqueness

Most of the results in the former section have in common that they can be proven by reducing the spin-glass problem which has an interaction decaying as $|i-j|^{-\alpha}$ to a non-random problem with an effective interaction which decays as $|i-j|^{-2\alpha}$. The proof for this non-random problem can then be at different levels of complication, dependent on the problem at hand. The reduction is performed by successively splitting off terms from the Hamiltonian and afterwards applying a Taylor expansion or a probabilistic estimate to this term. We can use Fubini's theorem to interchange the average over one disorder variable $\tilde{J}(i, j)$ and the thermal average with respect to the modified Hamiltonian $H_{i,j} \equiv H_0 + \tilde{J}(i, j)s_is_j$ where the term corresponding to this $\tilde{J}(i, j)$ has been subtracted. Because of condition (4) the final expression does not contain first-order terms, but has only terms of second and higher order in $\tilde{J}(i, j)$.

For example, for the free energy we use

$$E \ln \operatorname{tr} \exp[-(H_{i,j} + \tilde{J}(i, j)s_is_j)] = E \ln \operatorname{tr} \exp(-H_{i,j}) - (E\tilde{J}(i, j) \operatorname{tr} s_is_j \exp(-H(i, j)) =)0 + O(|i-j|^{-2\alpha}). \tag{7a}$$

(For a proof, see, for example [19, appendix].)

For the thermal expectations we use, if $\tilde{J}(i, j)$ is small,

$$E\frac{\operatorname{tr} f \exp[-(H_{i,j} + \tilde{J}(i, j)s_is_j)]}{\operatorname{tr} \exp[-(H_{i,j} + \tilde{J}(i, j)s_is_j)]} = E\frac{\operatorname{tr} f \exp(-H_{i,j})}{\operatorname{tr} \exp(-H_{i,j})} - \left(E\tilde{J}(i, j)\left(\frac{\operatorname{tr} s_is_jf \exp(-H_{i,j})}{\operatorname{tr} \exp(-H_{i,j})} - \frac{\operatorname{tr} f \exp(-H_{i,j})}{\operatorname{tr} \exp(-H_{i,j})}\,\frac{\operatorname{tr} s_is_j \exp(-H_{i,j})}{\operatorname{tr} \exp(-H_{i,j})}\right) = \right)0 + O(|i-j|^{-2\alpha}). \tag{7b}$$

(For a proof, see [18].)

For probabilistic estimates we use

$$E\frac{\operatorname{tr}\{\exp[-(H_{i,j} + \tilde{J}(i, j)s_is_j)]\chi_{\tilde{J}(i,j)s_is_j > c}\}}{\operatorname{tr} \exp[-(H_{i,j} + \tilde{J}(i, j)s_is_j)]} \leq \text{constant} \times \exp\left[-\left(\frac{c^2}{|i-j|^{-2\alpha}}\right)\right]. \tag{7c}$$

For a proof see [17] or [29].

This type of estimate often turns out to be useful if one wants to apply the Borel-Cantelli lemma.

The non-random part of the proof can be either known (subadditivity in theorem 1 [30], Araki's relative entropy method [13, 18, 31, 32], the Leuven energy-entropy inequalities [16, 33], the McBryan-Spencer estimates [19, 24, 34-6] to show the absence of symmetry breaking and upper bounds on correlation decay in one and two dimensions) or be developed for the problem at hand, like the block spin arguments of [17] and [20] which are used to map the system onto an effective high-T model (see also [22, 29, 37]) and the different polymer expansions of [20, 25, 26] (see also [22, 37]) which work in high-temperature regions.

The problem of weak against strong uniqueness comes in as follows. If one applies Fubini's theorem to the double integration with respect to the disorder variable $J(i, j)$ and the modified thermal average corresponding to $H_{i,j}$, this presupposes that $H_{i,j}$ does not contain any $J(i, j)$ dependence. In particular, $H_{i,j}$ contains boundary conditions and they should therefore be $J(i, j)$ independent for the proof to work. For example, let us consider the interaction energy W between left and right half-lines on Z, and consider the configuration to the right of the origin as the boundary condition.

If $\alpha > 1$, for each choice of this boundary condition the expression $W_\sigma(\{J\}, \{s\})$ is finite for each s and almost each $\{J\}$ (with respect to the J distribution) and so is the partition function [13]

$$Z(W_\sigma) = \mathrm{tr}_s \exp(W_\sigma(\{s\}, \{J\})). \tag{8}$$

However, if one allows J-dependent boundary conditions and takes the supremum of W_σ, over all boundary conditions σ, $\sup_\sigma W_\sigma(s) = \infty$ as soon as $\alpha < \frac{3}{2}$ [13]. The fact that $\sup_{s,\sigma} W_\sigma(s) < \infty$ for almost all J is the main ingredient in the strong uniqueness proof for the case $\alpha > \frac{3}{2}$ in [21], but as for the case $1 < \alpha < \frac{3}{2}$ one uses the estimates (7*a*)-(7*c*), in which we have used Fubini's theorem to suppress the 'bad' (large energy) configurations, one only obtains weak uniqueness.

A criterion for the absence of phase transitions is the disappearing of the Edwards-Anderson order parameter which is (formally) defined as [1]

$$Q_{\mathrm{EA}} = E\langle s_i\rangle_H^2. \tag{9}$$

By an ergodic theorem one can replace the average over the $\{J\}$ by a spatial average over the lattice. Weak uniqueness then implies that (J-almost surely)

$$Q_{\mathrm{EA}}^{(\mathrm{weak})} = \lim_{\Lambda\to\infty} \sum_{i\in\Lambda} \frac{\langle s_i\rangle_{\Lambda,\sigma}^2}{|\Lambda|} = 0 \tag{10a}$$

for each fixed boundary condition σ, or

$$\sup_\sigma \lim_{\Lambda\to\infty} E\langle s_i\rangle_{\Lambda,\sigma}^2 = 0.$$

Strong uniqueness means that [38] (J-almost surely)

$$Q_{\mathrm{EA}}^{(\mathrm{strong})} = \lim_{\Lambda\to\infty} \sup_\sigma \sum_{i\in\Lambda} \frac{\langle s_i\rangle_{\Lambda,\sigma}^2}{|\Lambda|} = 0. \tag{10b}$$

Expression (10*b*) is equivalent to a thermodynamic definition which uses a replicated system.

At present there are no examples known of spin-glass models on regular lattices which are weakly but not strongly unique. However, such behaviour does occur for certain temperatures in the Bethe lattice spin-glass model [39]. Of course, the Bethe lattice is somewhat pathological, as the size of the boundary is macroscopic and also the free energy depends on the boundary condition in the thermodynamic limit, but it shows at least in principle that the two notions are really different. A technically related problem occurs in unbounded spin systems where weak uniqueness corresponds to uniqueness of 'tempered' Gibbs states (see, for example, [40, 41]).

Summarising, we have reviewed some recent results on the absence of phase transitions for long-range spin-glass models, in particular in one dimension (a more heuristic treatment of this class of models can be found in [10]). We have discussed some common properties of their proofs and described the difference between 'weak' and 'strong' uniqueness.

Acknowledgments

I thank M Campanino, J Fröhlich, J L van Hemmen and E Olivieri for different collaborations on these problems. I gratefully acknowledge the hospitality of Professor

J T Lewis and the Dublin Institute for Advanced Studies, where this paper was written. I thank Professor O Penrose for inviting me to a stimulating conference at Heriot-Watt University and I thank the Lady Davis Foundation for financial support.

References

[1] Edwards S F and Anderson P W 1975 *J. Phys. F: Met. Phys.* **5** 965
[2] Binder K and Young A P 1986 *Rev. Mod. Phys.* **58** 801
[3] Chowdhury D 1986 *Spin Glasses and other Frustrated Systems* (Singapore: World Scientific)
[4] van Hemmen J L and Morgenstern I (ed) 1983 *Heidelberg Colloq. on Spin Glasses* (*Lecture Notes in Physics* **192**) (Berlin: Springer)
[5] van Hemmen J L and Morgenstern I (ed) 1987 *Heidelberg Colloq. on Glassy Dynamics* (*Lecture Notes in Physics* **275**) (Berlin: Springer)
[6] Bovier A and Fröhlich J 1986 *J. Stat. Phys.* **44** 347
[7] Fisher D S and Huse D 1986 *Phys. Rev. Lett.* **56** 1601; 1987 *J. Phys. A: Math. Gen.* **20** L997, L1005
[8] Bray A J and Moore M A 1987 *Phys. Rev. Lett.* **58** 57; 1987 *Heidelberg Colloq. on Glassy Dynamics* (*Lecture Notes in Physics* **275**) ed J L van Hemmen and L Morgenstern (Berlin: Springer) p 121
[9] Griffiths R B 1969 *Phys. Rev. Lett.* **23** 17
[10] Kotliar G, Anderson P W and Stein D L 1983 *Phys. Rev.* B **27** 602
[11] Khanin K M and Sinai Ya G 1979 *J. Stat. Phys.* **20** 573
[12] van Enter A C D and van Hemmen J L 1983 *J. Stat. Phys.* **32** 141
[13] van Enter A C D and van Hemmen J L 1985 *J. Stat. Phys.* **39** 1
[14] Goulart Rosa S 1982 *J. Phys. A: Math. Gen.* **15** L51
[15] Zegarlinsky B 1987 *J. Stat. Phys.* **47** 911
[16] Slegers L, Vansevenant A and Verbeure A 1985 *Phys. Lett.* **108A** 267
[17] Campanino M, Olivieri E and van Enter A C D 1987 *Commun. Math. Phys.* **108** 241
[18] van Enter A C D and Fröhlich J 1985 *Commun. Math. Phys.* **98** 425
[19] van Enter A C D 1985 *J. Stat. Phys.* **41** 315
[20] Campanino M and Olivieri E 1987 *Commun. Math. Phys.* **111** 555
[21] Khanin K M 1980 *Theor. Math. Phys.* **43** 445
[22] Cassandro M, Olivieri E and Tirozzi B 1982 *Commun. Math. Phys.* **87** 229
[23] Picco P 1983 *J. Stat. Phys.* **32** 627
[24] Picco P 1984 *J. Stat. Phys.* **36** 489
[25] Fröhlich J and Zegarlinsky B 1986 *Europhys. Lett.* **2** 53
[26] Fröhlich J and Zegarlinsky B 1987 *Commun. Math. Phys.* **110** 121
[27] Fröhlich J and Zegarlinsky B 1987 *Commun. Math. Phys.* **112** 553
[28] Aizenman M, Lebowitz J L and Ruelle D 1987 *Commun. Math. Phys.* **112** 3
[29] van Enter A C D 1987 *J. Stat. Phys.* **47** 905
[30] Griffiths R B 1968 *Phys. Rev.* **176** 655
[31] Araki H 1975 *Commun. Math. Phys.* **44** 1
[32] Fröhlich J and Pfister C E 1981 *Commun. Math. Phys.* **81** 277
[33] Fannes M, Vanheuverzwijn P and Verbeure A 1982 *J. Stat. Phys.* **29** 547
[34] McBryan O and Spencer T 1977 *Commun. Math. Phys.* **53** 298
[35] Messager A, Miracle-Sole A and Ruiz J 1984 *Ann. Inst. H Poincaré* **40** 85
[36] Ito K M 1982 *J. Stat. Phys.* **29** 747
[37] Cassandro M and Olivieri E 1981 *Commun. Math. Phys.* **80** 255
[38] van Enter A C D and Griffiths R B 1983 *Commun. Math. Phys.* **90** 319
[39] Chayes J T, Chayes L, Sethna J P and Thouless D J 1986 *Commun. Math. Phys.* **106** 41
[40] Campanino M, Capocaccia D and Olivieri E 1983 *J. Stat. Phys.* **33** 437
[41] Bellissard J and Hoegh-Krohn R 1982 *Commun. Math. Phys.* **84** 297